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The eight-vertex model and Painlevé VI

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Abstract
In this paper we establish a connection of Picard-type elliptic solutions of the Painlevé VI equation with the special solutions of the non-stationary Lamé equation. The latter appeared in the study of the ground-state properties of Baxter's solvable eight-vertex lattice model at a particular point, $\eta = \pi/3$, of the disordered regime.

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1. Introduction

The Painlevé transcendents have numerous remarkable applications in the theory of integrable models of statistical mechanics and quantum field theory (see, for instance, [1–3]). We mention, in particular, the calculation of the 'supersymmetric index' and related problems of dilute polymers on a cylinder which lead to Painlevé III [4, 5]. These problems are connected with the finite volume massive sine-Gordon model with $N = 2$ supersymmetry. The lattice analogue of this continuous quantum field theory corresponds to a special case of Baxter's famous solvable eight-vertex lattice model [6]. In this paper we continue our study [7] of this special model on a finite lattice and unravel its deep connections with Painlevé VI theory.

We consider the eight-vertex model on a square lattice with an odd number, $N = 2n + 1$, of columns and periodic boundary conditions. The eigenvalues of the row-to-row transfer matrix of the model, $T(u)$, satisfy the TQ equation [6]

$$T(u)Q(u) = \phi(u-\eta)Q(u+2\eta) + \phi(u+\eta)Q(u-2\eta), \tag{1}$$

where u is the spectral parameter,

$$\phi(u) = \vartheta_1^N(u \mid \mathsf{q}), \qquad \mathsf{q} = \mathrm{e}^{\mathrm{i}\pi\tau}, \qquad \operatorname{Im}\tau > 0, \tag{2}$$

and $\vartheta_1(u \mid q)$ is the standard theta-function with the periods π and $\pi\tau$ (we follow the notation of [8]). Here we consider a special case $\eta = \pi/3$, where the ground-state eigenvalue is known [9, 10] to have a very simple form for all (odd) N

$$T(u) = \phi(u), \qquad \eta = \frac{\pi}{3}. \tag{3}$$

Equation (1) with this eigenvalue, $T(u)$, has two different solutions [11, 12], $Q_\pm(u) \equiv Q_\pm(u, q, n)$, which are entire functions of the variable u and obey the following periodicity conditions [6, 13]:[1]

$$\begin{aligned} &Q_\pm(u+\pi) = \pm(-1)^n Q_\pm(u), \\ &Q_\pm(u+\pi\tau) = \mathsf{q}^{-N/2}\, \mathrm{e}^{-\mathrm{i}Nu}\, Q_\mp(u), \\ &Q_\pm(-u) = Q_\pm(u). \end{aligned} \tag{4}$$

The above requirements uniquely determine $Q_\pm(u)$ to within a common u-independent normalization factor. It is convenient to rewrite the functional equation (1) for $Q_\pm(u)$ in the form

$$\Psi_\pm(u) + \Psi_\pm\left(u + \frac{2\pi}{3}\right) + \Psi_\pm\left(u + \frac{4\pi}{3}\right) = 0, \tag{5}$$

where

$$\Psi_\pm(u) \equiv \Psi_\pm(u, \mathsf{q}, n) = \frac{\vartheta_1^{2n+1}(u \mid \mathsf{q})}{\vartheta_1^n(3u \mid \mathsf{q}^3)} Q_\pm(u, \mathsf{q}, n) \tag{6}$$

are the meromorphic functions of the variable u for any fixed values of q and n. As shown in [7], these functions satisfy the non-stationary Lamé equation

$$6\mathsf{q}\frac{\partial}{\partial \mathsf{q}}\Psi(u, \mathsf{q}, n) = \left\{-\frac{\partial^2}{\partial u^2} + 9n(n+1)\wp(3u \mid \mathsf{q}^3) + c(\mathsf{q}, n)\right\}\Psi(u, \mathsf{q}, n), \tag{7}$$

where the elliptic Weierstrass $\wp$-function, $\wp(v \mid \mathrm{e}^{\mathrm{i}\pi\epsilon})$, has the periods π and $\pi\epsilon$ [8]. The constant $c(\mathsf{q}, n)$ appearing in (7) is totally controlled by the (u-independent) normalization of $Q_\pm(u)$. Note that this equation (in fact, a more general equation, usually called the non-autonomous Lamé equation) arose previously [14–17] in different contexts. We will not explore these connections here.

Let us quote other relevant results of [7]. Define the combinations

$$Q_1(u) = (Q_+(u) + Q_-(u))/2, \qquad Q_2(u) = (Q_+(u) - Q_-(u))/2 \tag{8}$$

such that

$$Q_{1,2}(u+\pi) = (-1)^n Q_{2,1}(u). \tag{9}$$

Bearing in mind this simple relation we will only consider $Q_1(u)$, writing it as $Q_1^{(n)}(u)$ to indicate the n-dependence. Obviously the linear relations (5) and (7) remain unaffected if $Q_\pm(u)$ in (6) is replaced by $Q_1^{(n)}(u)$. The partial differential equation (7) has, of course, many solutions. Here we are only interested in the very special solutions, relevant to our original problem of the eight-vertex model (by definition the functions $Q_1^{(n)}(u)$ are entire quasi-periodic functions of u with the periods π and $2\pi\tau$). Introduce new variables γ and x, instead of q and u,

$$\gamma \equiv \gamma(\mathsf{q}) = -\left[\frac{\vartheta_1(\pi/3 \mid \mathsf{q}^{1/2})}{\vartheta_2(\pi/3 \mid \mathsf{q}^{1/2})}\right]^2, \qquad x = \gamma\left[\frac{\vartheta_3(u/2 \mid \mathsf{q}^{1/2})}{\vartheta_4(u/2 \mid \mathsf{q}^{1/2})}\right]^2, \tag{10}$$

and new functions $\mathcal{P}_n(x, z)$ instead of $Q_1^{(n)}(u)$,

$$Q_1^{(n)}(u) = \mathcal{N}(\mathsf{q}, n)\vartheta_3(u/2 \mid \mathsf{q}^{1/2})\vartheta_4^{2n}(u/2 \mid \mathsf{q}^{1/2})\mathcal{P}_n(x, z), \qquad z = \gamma^{-2}, \tag{11}$$

where $\mathcal{N}(\mathsf{q}, n)$ is an arbitrary normalization factor (which remains at our disposal). The properties of the functions $\mathcal{P}_n(x, z)$ corresponding to the required solutions of (5) and (7) can be summarized by the following

[1] The factor $(-1)^n$ in (4) and (9) reflects our convention for labelling the eigenvalues for different n.

Conjecture 1.

(a) *The functions $\mathcal{P}_n(x,z)$ are polynomials in x, z of the degree n in x,*

$$\mathcal{P}_n(x,z) = \sum_{k=0}^{n} r_k^{(n)}(z) x^k, \tag{12}$$

while $r_i^{(n)}(z)$, $i = 0, \ldots, n$, are polynomials in z of the degree

$$\deg\left[r_k^{(n)}(z)\right] \leqslant \lfloor n(n-1)/4 + k/2 \rfloor \tag{13}$$

with positive integer *coefficients. The normalization of $\mathcal{P}_n(x,z)$ is fixed by the requirement $r_n^{(n)}(0) = 1$ and $\lfloor x \rfloor$ denotes the largest integer not exceeding x.*

(b) *The coefficients of the lowest and highest powers in x, corresponding to $k = 0$ and $k = n$ in (12), read*

$$r_0^{(n)}(z) = \tau_n(z, -1/3), \qquad r_n^{(n)}(z) = \tau_{n+1}(z, 1/6), \tag{14}$$

where the functions $\tau_n(z,\xi)$ (for each fixed value of the their second argument ξ) are determined by the recurrence relation

$$\begin{aligned} &2z(z-1)(9z-1)^2[\log \tau_n(z)]''_z + 2(3z-1)^2(9z-1)[\log \tau_n(z)]'_z \\ &\quad + 8\left[2n - 4\xi - \frac{1}{3}\right]^2 \frac{\tau_{n+1}(z)\tau_{n-1}(z)}{\tau_n^2(z)} - [12(3n - 6\xi - 1)(n - 2\xi) \\ &\quad + (9z-1)(n-1)(5n - 12\xi)] = 0, \end{aligned} \tag{15}$$

with the initial condition

$$\tau_0(z,\xi) = 1, \qquad \tau_1(z,\xi) = -4\xi + 5/3. \tag{16}$$

The functions $\tau_n(z,\xi)$ are polynomials in z for all $n = 0, 1, 2, \ldots, \infty$.

The conjecture has been verified by an explicit calculation [7] of $\mathcal{P}_n(x,z)$ for all $n \leqslant 50$.

The polynomials $\mathcal{P}_n(x,z)$ can be effectively calculated using the algebraic form (see [7]) of equation (7). The first few of them read[2]

$$\mathcal{P}_0(x,z) = 1, \qquad \mathcal{P}_1(x,z) = x + 3, \qquad \mathcal{P}_2(x,z) = x^2(1+z) + 5x(1+3z) + 10, \tag{17}$$

$$\mathcal{P}_3(x,z) = x^3(1+3z+4z^2) + 7x^2(1+5z+18z^2) + 7x(3+19z+18z^2) + 35 + 21z. \tag{18}$$

As explained in [7], equation (7) leads to a descending recurrence relation for the coefficients in (12), in the sense that each coefficient $r_k^{(n)}(z)$ with $k < n$ can be recursively calculated in terms of $r_m^{(n)}(z)$, with $m = k+1, \ldots, n$ and, therefore, can eventually be expressed through the coefficient $r_n^{(n)}(z)$ of the leading power of x. The conditions that this procedure truncates (and thus defines a polynomial, but not an infinite series in negative powers of x) completely determine the starting leading coefficient as a function of z. The above conjecture implies that these truncation conditions are equivalent to the particular case ($\xi = 1/6$) of recurrence relations (15), (16). A similar statement applies to the (smallest power in x) coefficient $r_0^{(n)}$ of $\mathcal{P}_n(x,z)$. The results quoted above are due to [7] (except for the expression (14) for $r_0^{(n)}$, which is new).

In this paper we show that the recurrence relation (15) exactly coincides with that for the tau-functions associated with special elliptic solutions of the Painlevé VI equation, revealing hitherto unknown connections of the eight-vertex model and the non-stationary Lamé equation (7) to Painlevé VI theory.

[2] The higher polynomials with $n \leqslant 50$ are available in electronic form upon request to the authors.

Before concluding this introduction, let us briefly mention some problems related to the special eight-vertex model considered here. In the trigonometric limit ($q \to 0$) this model reduces to the special six-vertex model (with the parameter $\Delta = -1/2$), which is closely connected with various interesting combinatorial problems [18–21], particularly, with the theory of alternating sign matrices. The fact that the polynomials $\mathcal{P}_n(x,z)$ have *positive integer coefficients* makes it plausible to suggest that these coefficients have some (yet unknown) combinatorial interpretation.

In the scaling limit

$$n \to \infty, \qquad \mathsf{q} \to 0, \qquad t = 8n\mathsf{q}^{3/2} = \text{fixed}, \tag{19}$$

functions (6) essentially reduce to the ground-state eigenvalues $\mathcal{Q}_\pm(\theta) \equiv \mathcal{Q}_\pm(\theta, t)$ of the **Q**-operators [11, 22] of the restricted massive sine-Gordon model (at the so-called, supersymmetric point) on a cylinder of the spatial circumference R, where $t = MR$ and M is the soliton mass and the variable θ is defined as $u = \pi\tau/2 - \mathrm{i}\theta/3$. With a suitable t-dependent normalization of $\mathcal{Q}_\pm(\theta)$, equation (7) then reduces to the 'non-stationary Mathieu equation' [7],

$$t\frac{\partial}{\partial t}\mathcal{Q}_\pm(\theta,t) = \left\{\frac{\partial^2}{\partial\theta^2} - \frac{1}{8}t^2(\cosh 2\theta - 1)\right\}\mathcal{Q}_\pm(\theta,t). \tag{20}$$

This equation determines the asymptotic behaviour of $\mathcal{Q}_\pm(\theta)$ at large θ

$$\log\mathcal{Q}_\pm(\theta) = -\tfrac{1}{4}t\,\mathrm{e}^{\theta} + \log\mathcal{D}_\pm(t) + 2\,(\partial_t\log\mathcal{D}_\pm(t) - t/8)\,\mathrm{e}^{-\theta} + O(\mathrm{e}^{-2\theta}), \qquad \theta \to +\infty, \tag{21}$$

where $\mathcal{D}_\pm(t)$ are the Fredholm determinants, which previously appeared in connection with the 'supersymmetric index' and the dilute polymers on a cylinder [4, 5, 23, 24]. Note, in particular, that the quantity

$$F(t) = \frac{\mathrm{d}}{\mathrm{d}t}U(t), \qquad U(t) = \log\frac{\mathcal{D}_+(t)}{\mathcal{D}_-(t)}, \tag{22}$$

describes the free energy of a single incontractible polymer loop and satisfies the Painlevé III equation [23]

$$\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}t\frac{\mathrm{d}}{\mathrm{d}t}U(t) = \frac{1}{2}\sinh 2U(t). \tag{23}$$

2. Painlevé VI equation

The Painleve $\mathbf{P}_{\mathrm{VI}}(\alpha,\beta,\gamma,\delta)$ is the following second-order differential equation [25, 26],

$$q''(t) = \frac{1}{2}\left(\frac{1}{q(t)} + \frac{1}{q(t)-1} + \frac{1}{q(t)-t}\right)q'(t)^2 - \left(\frac{1}{t} + \frac{1}{(t-1)} + \frac{1}{q(t)-t}\right)q'(t) + \frac{q(t)(q(t)-1)(q(t)-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{q(t)^2} + \gamma\frac{t-1}{(q(t)-1)^2} + \delta\frac{t(t-1)}{(q(t)-t)^2}\right], \tag{24}$$

where the following parameterizations of four constants [27] are chosen

$$\alpha = \tfrac{1}{2}\kappa_\infty^2, \qquad \beta = -\tfrac{1}{2}\kappa_0^2, \qquad \gamma = \tfrac{1}{2}\kappa_1^2, \qquad \delta = \tfrac{1}{2}(1-\theta^2), \tag{25}$$

$$\kappa_0 = b_1 + b_2, \qquad \kappa_1 = b_1 - b_2, \qquad \kappa_\infty = b_3 - b_4, \qquad \theta = b_3 + b_4 + 1. \tag{26}$$

This equation is equivalent to the Hamiltonian system $H_{\mathrm{VI}}(t;q,p)$ described by the equations

$$\frac{\mathrm{d}q}{\mathrm{d}t}=\frac{\partial H}{\partial p},\qquad \frac{\mathrm{d}p}{\mathrm{d}t}=-\frac{\partial H}{\partial q},\tag{27}$$

with the Hamiltonian function

$$\begin{aligned}H_{\mathrm{VI}}(t;q,p)=\frac{1}{t(t-1)}&[q(q-1)(q-t)p^2\\&-\{\kappa_0(q-1)(q-t)+\kappa_1 q(q-t)+(\theta-1)q(q-1)\}p+\kappa(q-t)],\end{aligned}\tag{28}$$

where $q\equiv q(t)$, $p\equiv p(t)$ and

$$\kappa=\tfrac{1}{4}(\kappa_0+\kappa_1+\theta-1)^2-\tfrac{1}{4}\kappa_\infty^2.\tag{29}$$

One can introduce an auxiliary Hamiltonian $h(t)$,

$$h(t)=t(t-1)H(t)+e_2(b_1,b_3,b_4)t-\tfrac{1}{2}e_2(b_1,b_2,b_3,b_4),\tag{30}$$

where $e_i(x_1,\ldots,x_n)$ is the ith elementary symmetric function in n variables.

Okamoto [27] showed that, for each pair $\{q(t),p(t)\}$ satisfying (27), the function $h(t)$ solves the $\mathbf{E}_{\mathrm{VI}}$ equation,

$$h'(t)[t(1-t)h''(t)]^2+[h'(t)[2h(t)-(2t-1)h'(t)]+b_1b_2b_3b_4]^2=\prod_{k=1}^{4}\left(h'(t)+b_k^2\right),\tag{31}$$

and $q(t)$ solves $\mathbf{P}_{\mathrm{VI}}(\alpha,\beta,\gamma,\delta)$, (24).

Conversely, for each solution $h(t)$ of (31), such that $\frac{\mathrm{d}^2}{\mathrm{d}t^2}h(t)\neq 0$, there exists a solution $\{q(t),p(t)\}$ of (27), where $q(t)$ solves (24). An explicit correspondence between the three sets $\{q(t),q'(t)\}$, $\{q(t),p(t)\}$ and $\{h(t),h'(t),h''(t)\}$ is given by birational transformations, which can be found in [27].

The $\mathbf{E}_{\mathrm{VI}}$ equation is a particular case of a more general class of the second-order second-degree equation

$$y''(t)^2=F(t,y(t),y'(t)),\tag{32}$$

where F is rational in $y(t)$, $y'(t)$, locally analytic in t, with the property that the only movable singularities of $y(t)$ are poles. Such equations were classified in [28], where equation (31) is referred to the 'SD-I type'. Originally, we have obtained equations of the form (32) for asymptotics of the polynomals $\mathcal{P}_n(x,z)$ and only then reduced them to (31) and (24).

The group of Backlund transformations of $\mathbf{P}_{\mathrm{VI}}$ is isomorphic to the affine Weyl group of the type D_4. It contains the following transformations of parameters (only 5 of them are independent)

$$w_1:b_1\leftrightarrow b_2,\qquad w_2:b_2\leftrightarrow b_3,\qquad w_3:b_3\leftrightarrow b_4,\qquad w_4:b_3\to -b_3,b_4\to -b_4,\tag{33}$$

$$x^1:\kappa_0\leftrightarrow\kappa_1,\qquad x^2:\kappa_0\leftrightarrow\kappa_\infty,\qquad x^3:\kappa_0\leftrightarrow\theta,\tag{34}$$

and the parallel transformation

$$l_3:\mathbf{b}\equiv(b_1,b_2,b_3,b_4)\to\mathbf{b}^+\equiv(b_1,b_2,b_3+1,b_4).\tag{35}$$

Here we shall use only two transformations: x^2 and l_3. The canonical transformation x^2 corresponds to a simple change of variables [27],

$$x_\star^2:(q,p,H,t)\to(q^{-1},\epsilon q-q^2p,-H/t^2,1/t),\qquad \epsilon=\tfrac{1}{2}(\kappa_0+\kappa_1+\theta-1+\kappa_\infty).\tag{36}$$

The birational canonical transformation,

$$\{q, p\} = \{q(\mathbf{b}), p(\mathbf{b})\} \to \{q^+, p^+\} = \{q(\mathbf{b}^+), p(\mathbf{b}^+)\}, \tag{37}$$

corresponding to l_3, can be obtained from the observation that [27]

$$h^+(t, q^+, p^+) = h(t, q, p) - q(q-1)p + (b_1+b_4)q - \tfrac{1}{2}(b_1+b_2+b_4). \tag{38}$$

Here we shall only give the expression for q in terms of $\{q^+, p^+\}$:

$$\begin{aligned} q\big\{&p_+^2 q_+(q_+-1)(q_+-t) + p_+[2(1+b_1+b_3)q_+(1-q_+) + (b_1+b_2)(q_+-t) + 2b_1(t-1)q_+] \\ &+ (1+b_1+b_3)[(1+b_1+b_3)(q_+-1) + t(1+b_3-b_1) + b_1 - b_2]\big\} \\ &- t[p_+(q_+-1) - 1 - b_1 - b_3][p_+q_+(q_+-1) - (1+b_1+b_3)q_+ + b_1 + b_2] = 0. \end{aligned} \tag{39}$$

Later we will need the transformation $x^2 l_3 x^2$

$$x^2 l_3 x^2 : \{\mathbf{b}\} = \{b_1, b_2, b_3, b_4\} \to \{\tilde{\mathbf{b}}\} = \left\{b_1 + \tfrac{1}{2}, b_2 + \tfrac{1}{2}, b_3 + \tfrac{1}{2}, b_4 + \tfrac{1}{2}\right\}. \tag{40}$$

Combining (36)–(39) one obtains

$$\begin{aligned} q = \{&\tilde{p}^2(\tilde{q}-t)(\tilde{q}-1) + \tilde{p}(2+b_1+b_2+b_3+b_4+(b_1-b_2)t - (2+2b_1+b_3+b_4)\tilde{q}) \\ &+ (1+b_1+b_3)(1+b_1+b_4)\}/\{(1+b_1+b_3+\tilde{p}(1-\tilde{q}))(1+b_1+b_4+\tilde{p}(1-\tilde{q}))\}, \end{aligned} \tag{41}$$

where $q = q(\mathbf{b})$ and $\tilde{q} = q(\tilde{\mathbf{b}})$, $\tilde{p} = p(\tilde{\mathbf{b}})$.

3. Special elliptic solutions

It goes back to Picard [29] that if parameters (26) satisfy

$$b_1 = b_2 = 0, \qquad b_3 = b_4 = -1/2 \tag{42}$$

then a general solution of $\mathbf{P}_{\rm VI}$ reads (see also [30])

$$q(t) = \wp(c_1\omega_1 + c_2\omega_2; \omega_1, \omega_2) + \frac{t+1}{3}, \tag{43}$$

where $\wp(u; \omega_1, \omega_2)$ is the Weierstrass elliptic function with half-periods $\omega_{1,2}$, $c_{1,2}$ are arbitrary constants and $\omega_{1,2}(t)$ are two linearly independent solutions of the hypergeometric equations

$$t(1-t)\omega''(t) + (1-2t)\omega'(t) - \tfrac{1}{4}\omega(t) = 0. \tag{44}$$

It is convenient to choose

$$\omega_1(t) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right) = \mathbf{K}(t^{1/2}), \qquad \omega_2(t) = \mathrm{i}\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-t\right) = \mathrm{i}\mathbf{K}'(t^{1/2}), \tag{45}$$

where $\mathbf{K}$ and $\mathbf{K}'$ are the complete elliptic integrals of the modulus $k = t^{1/2}$.

Using expressions for invariants of the Weierstrass function e_1, e_2, e_3,

$$e_1 = 1 - \frac{t+1}{3}, \qquad e_2 = t - \frac{t+1}{3}, \qquad e_3 = -\frac{t+1}{3}, \tag{46}$$

we can rewrite Picard's solution of $\mathbf{P}_{\rm VI}$ as

$$q(t) = \mathrm{ns}^2(c_1\mathbf{K} + c_2\mathrm{i}\mathbf{K}', k), \qquad k = t^{1/2}. \tag{47}$$

Let us choose

$$c_1 = 1, \qquad c_2 = \tfrac{1}{3} \tag{48}$$

and denote corresponding solution of (24) as $q_1(t)$. Using addition theorems for elliptic functions it is easy to show that $q_1(t)$ satisfies the algebraic equation

$$q_1^4(t) - 4tq_1^3(t) + 6tq_1^2(t) - 4tq_1(t) + t^2 = 0. \tag{49}$$

In fact, this algebraic solution of $\mathbf{P}_{\rm VI}$ is very special. It solves (24) not only for Picard's choice of parameters (42) but also for

$$b_1 = \xi - \tfrac{1}{6}, \qquad b_2 = 0, \qquad b_3 = \xi - \tfrac{2}{3}, \qquad b_4 = 2\xi - \tfrac{5}{6}, \tag{50}$$

where ξ is an arbitrary parameter. This happens because $\mathbf{P}_{\rm VI}$, (24), splits up into two different equations which are both satisfied by (49).

Using the above formulae we can easily find expressions for $p_1(t)$ and $h_1(t)$ corresponding to (49):

$$\begin{aligned} p_1(t) &= \frac{(1-3\xi)\big(3t - 2tq_1(t) - q_1^2(t)\big)}{6t(q_1(t)-1)^2}, \\ h_1(t) &= \frac{1-2t}{72} + \frac{(1-2\xi)(1-3\xi)}{4}\frac{(t + q_1(t) - 2tq_1(t))}{q_1(t) - t}. \end{aligned} \tag{51}$$

Now we shall assume that this solution corresponds to the case $n = 1$ and apply Backlund transformation $x^2(l_3)^{1-n}x^2$ to obtain a series of solutions $\{q_n(t), p_n(t), h_n(t)\}$ with parameters

$$b_1 = \frac{1}{3} + \xi - \frac{n}{2}, \qquad b_2 = \frac{1}{2} - \frac{n}{2}, \qquad b_3 = -\frac{1}{6} + \xi - \frac{n}{2}, \qquad b_4 = -\frac{1}{3} + 2\xi - \frac{n}{2}. \tag{52}$$

At this stage we are ready to establish a connection with a parameterization from the previous sections. Let us assume that the elliptic nome q in (2), (6), (7) and (10) is given by

$$\mathsf{q} = \exp\left\{\mathrm{i}\pi\frac{2}{3}\frac{\mathbf{K}'(k)}{\mathbf{K}(k)}\right\}, \qquad k = t^{1/2}, \tag{53}$$

where $\mathbf{K}(k)$ and $\mathbf{K}'(k)$ are defined by (45). Using Landen transformation for elliptic functions it is easy to obtain the following rational parameterization for $z = 1/\gamma(\mathsf{q})^2$ defined in (10), (11), and for t, $q_1(t)$ in (49),

$$z = \frac{1}{\gamma^2(\mathsf{q})} = \frac{1+\overline{\gamma}}{(3-\overline{\gamma})\overline{\gamma}}, \qquad t = \frac{(1-\overline{\gamma})(3+\overline{\gamma})^3}{(1+\overline{\gamma})(3-\overline{\gamma})^3}, \qquad q_1(t) = \frac{(1-\overline{\gamma})(3+\overline{\gamma})}{(1+\overline{\gamma})(3-\overline{\gamma})}, \tag{54}$$

in terms of a new parameter $\bar{\gamma} \equiv \gamma(\mathsf{q}^{1/2})$, defined by (10) with q replaced by $\mathsf{q}^{1/2}$. Note that such parameterization of Picard's solutions of $\mathbf{P}_{\rm VI}$ with the above choice (48) of the parameters c_1 and c_2 has already appeared in [30, 31].

From these formulae one can get an explicit connection of variables t and z:

$$t = \frac{(z-1)(1-9z)^3}{32z}\left[1 + \frac{27z^2 - 18z - 1}{\sqrt{(1-z)(1-9z)^3}}\right]. \tag{55}$$

Now we can construct a sequence of τ-functions associated with a series of auxiliary Hamiltonians $h_n(t)$. It appears that corresponding τ-functions are polynomials in variable z.

First, let us introduce a sequence of functions $\sigma_n(z)$ considering them as functions of z

$$\begin{aligned} \sigma_n(z) = \frac{1}{tz}\sqrt{\frac{1-9z}{1-z}}\Bigg\{ & h_n(t) + \frac{1}{72}(2t-1) + (n-1)^2\left[\frac{t-1}{4} + \frac{1-9z}{8}\sqrt{(1-t)z}\right] \\ & + (n-1)\left(\xi - \frac{5}{12}\right)\left[1 - t + \frac{t(1-3z)}{\sqrt{(1-z)(1-9z)}}\right] \\ & + \left(\xi - \frac{1}{2}\right)\left(\xi - \frac{1}{3}\right)\left[\frac{3}{2} - \sqrt{\frac{1-t}{z}}\right]\Bigg\}. \end{aligned} \tag{56}$$

Comparing it with (51) and using (54) it is easy to see that

$$\sigma_1(z) = 0. \tag{57}$$

Then using Backlund transformations $x^2 l_3^{-1} x^2$ and $x^2 l_3 x^2$ it is not difficult to show that

$$\sigma_i(z) = 0, \qquad i = 0, 1, 2. \tag{58}$$

The easiest way to do that is to calculate $h_i(t)$, $i = 0, 2$, in terms of $h_1(t)$ (51), substitute into (56) and use a rational parameterization (54).

Now let us introduce a family of τ-functions $\tau_n(z, \xi)$ via

$$\sigma_n(z) = \frac{\mathrm{d}}{\mathrm{d}z}[\log \tau_n(z, \xi)] \tag{59}$$

and fix a normalization for $n = 0, 1, 2$ as

$$\tau_0(z, \xi) = 1, \qquad \tau_1(z, \xi) = -4\xi + 5/3, \qquad \tau_2(z, \xi) = 3(2\xi - 1)(3\xi - 1). \tag{60}$$

Using Okamoto's Toda-recursion relations for τ-functions for $\mathbf{P}_{\mathrm{VI}}$, generated via successive applications of parallel transformation l_3 [27], one can show that the recurrence relation for $\tau_n(z, \xi)$ exactly coincides with (15). Thus, we showed that the leading coefficient and the constant term of $\mathcal{P}_n(x, z)$ (considered as polynomials in x) can be expressed in terms τ-functions for special solutions of $\mathbf{P}_{\mathrm{VI}}$.

At the moment we do not have a complete proof of the polynomiality of $\tau_n(z)$. Note that this property takes place provided that two successive τ-functions $\tau_n(z)$ and $\tau_{n+1}(z)$ do not have a non-trivial common divisor (which is a polynomial in z).

One of the challenging problems is to find a determinant representation for $\tau_n(z, \xi)$ similar to the known other polynomial solutions of the Painlevé equations. It could help to clarify the structure of $\mathcal{P}_n(x, z)$ and possibly to establish a connection with problems of combinatorics.

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